

## 5. Finite Difference Method

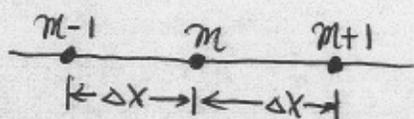
### 5.1. Basic Concept.

#### 5.1.1. Finite-Difference Representation of derivatives.

Consider 1D heat conduction equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

\* Spatial domain



At any time  $t = pat$ ,

The temperature at point  $m$  is related to the temperature at  $m+1$ :

$$[\text{Eq. 5.1}] T_{m+1}^p = T_m^p + \left(\frac{\partial T}{\partial x}\right)_m^p \Delta x + \frac{1}{2!} \left(\frac{\partial^2 T}{\partial x^2}\right)_m^p \Delta x^2 + \frac{1}{3!} \left(\frac{\partial^3 T}{\partial x^3}\right)_m^p \Delta x^3 + \dots$$

(forward Taylor series)

The temperature at point  $m$  is related to the temperature at  $m-1$ :

$$[\text{Eq. 5.2}] T_{m-1}^p = T_m^p - \left(\frac{\partial T}{\partial x}\right)_m^p \Delta x + \frac{1}{2!} \left(\frac{\partial^2 T}{\partial x^2}\right)_m^p \Delta x^2 - \frac{1}{3!} \left(\frac{\partial^3 T}{\partial x^3}\right)_m^p \Delta x^3 + \dots$$

(backward Taylor series)

### First Derivative Approximations.

(1) forward difference, (from Eq. 5.1)

$$\left(\frac{\partial T}{\partial x}\right)_m^p = \frac{T_{m+1}^p - T_m^p}{\Delta x} \quad O(\Delta x)$$

(2) backward difference: (from Eq. 5.2)

$$\left(\frac{\partial T}{\partial x}\right)_m^p = \frac{T_m^p - T_{m-1}^p}{\Delta x} \quad O(\Delta x)$$

(3) Central difference: (Eq. 5.1 - Eq. 5.2)

$$\left(\frac{\partial T}{\partial x}\right)_m^p = \frac{T_{m+1}^p - T_{m-1}^p}{2\Delta x} \quad O(\Delta x^2)$$

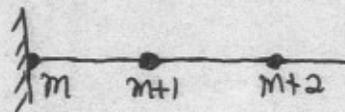
### Second Derivative Approximation.

(1) Central difference: (Eq. 5.1 + Eq. 5.2)

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_m^p = \frac{T_{m+1}^p - 2T_m^p + T_{m-1}^p}{(\Delta x)^2} \quad O(\Delta x^2) \quad \checkmark$$

### Treatment of Boundaries

(1) Left boundary

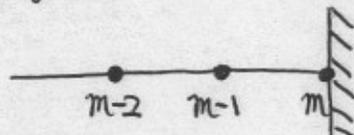


$$\text{[Eq. 5.3]} \quad T_{m+2}^p = T_m^p + \left(\frac{\partial T}{\partial x}\right)_m^p (2\Delta x) + \frac{1}{2!} \left(\frac{\partial^2 T}{\partial x^2}\right)_m^p (2\Delta x)^2 + \dots$$

$$\left(\frac{\partial T}{\partial x}\right)_m^p = \frac{-3T_m^p + 4T_{m+1}^p - T_{m+2}^p}{2\Delta x} \quad O(\Delta x^2)$$

$$(\text{Eq. 5.3} - 4 \times \text{Eq. 5.1})$$

(2) Right boundary



$$\text{[Eq. 5.4]} \quad T_{m-2}^p = T_m^p - \left(\frac{\partial T}{\partial x}\right)_m^p (2\Delta x) + \frac{1}{2!} \left(\frac{\partial^2 T}{\partial x^2}\right)_m^p (2\Delta x)^2 + \dots$$

$$\left(\frac{\partial T}{\partial x}\right)_m^p = \frac{3T_m^p - 4T_{m-1}^p + T_{m-2}^p}{2\Delta x} \quad O(\Delta x^2)$$

$$(\text{Eq. 5.4} - 4 \times \text{Eq. 5.2})$$

\* Time domain

Forward difference expression:

$$\left(\frac{\partial T}{\partial t}\right)_m^p = \frac{T_m^{p+1} - T_m^p}{\Delta t} \quad O(\Delta t) \quad \checkmark$$

5.1.2. Finite-Difference Representation of Heat Conduction Equation.

\* Explicit scheme: (1D)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} :$$

$$\frac{T_m^{p+1} - T_m^p}{\Delta t} = \alpha \frac{T_{m+1}^p - 2T_m^p + T_{m-1}^p}{\Delta x^2} \quad O(\Delta t, \Delta x^2)$$

\* Implicit scheme: (1D)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} :$$

$$\frac{T_m^{p+1} - T_m^p}{\Delta t} = \alpha \frac{T_{m+1}^{p+1} - 2T_m^{p+1} + T_{m-1}^{p+1}}{\Delta x^2} \quad O(\Delta t, \Delta x^2)$$

\* Crank-Nicolson Scheme: (1D)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} :$$

$$\frac{T_m^{p+1} - T_m^p}{\Delta t} = \frac{\alpha}{2} \left[ \frac{T_{m+1}^p - 2T_m^p + T_{m-1}^p}{\Delta x^2} + \frac{T_{m+1}^{p+1} - 2T_m^{p+1} + T_{m-1}^{p+1}}{\Delta x^2} \right] \quad O(\Delta t^2, \Delta x^2)$$

\* Explicit scheme: (2D)

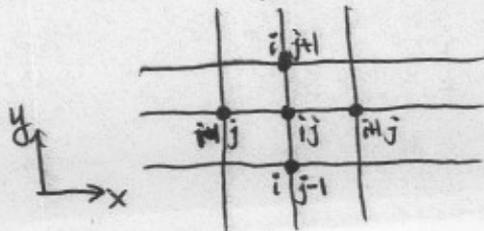
$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) :$$

$$\frac{T_{ij}^{p+1} - T_{ij}^p}{\Delta t} = \alpha \left[ \frac{T_{i+1,j}^p - 2T_{ij}^p + T_{i-1,j}^p}{\Delta x^2} + \frac{T_{i,j+1}^p - 2T_{ij}^p + T_{i,j-1}^p}{\Delta y^2} \right]$$

\* Implicit scheme: (2D)

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) :$$

$$\frac{T_{ij}^{p+1} - T_{ij}^p}{\Delta t} = \alpha \left[ \frac{T_{i+1,j}^{p+1} - 2T_{ij}^{p+1} + T_{i-1,j}^{p+1}}{\Delta x^2} + \frac{T_{i,j+1}^{p+1} - 2T_{ij}^{p+1} + T_{i,j-1}^{p+1}}{\Delta y^2} \right]$$



## 5.2. Stability Analysis

\* Consider 1D explicit scheme of transient heat conduction equation.

$$\frac{T_m^{p+1} - T_m^p}{\Delta t} = \alpha \frac{T_{m+1}^p - 2T_m^p + T_{m-1}^p}{\Delta x^2}$$

$$\text{i.e.: } T_m^{p+1} = T_m^p + \frac{1}{M} (T_{m+1}^p - 2T_m^p + T_{m-1}^p)$$

$$\text{where: } \underline{M}^{-1} \equiv \frac{\alpha \Delta t}{\Delta x^2} \text{ (dimensionless)}$$

Stability analysis is concerned with the growth of errors while computation is performed according to certain finite difference scheme. For an unstable scheme, the error will grow without bound as computation continues.

Note: The numerical solution of the problem can be considered to be the sum of the exact solution (infinite accuracy) and an error term,  $\epsilon_m^p$  (corresponding to  $T_m^p$ ).

To examine the propagation/growth of errors as time increases, we consider a Fourier mode of the error:

$$\underline{\epsilon_m^p} = e^{i\alpha p \Delta t} \cdot e^{i\beta m \Delta x}$$

( $\alpha$  is in general a complex number)

$$\text{or: } \underline{\epsilon_m^p} = G^p \cdot e^{i\beta m \Delta x}$$

( $G$ : amplification factor)

$$\begin{aligned}
 \text{So: } \varepsilon_m^{p+1} &= \varepsilon_m^p + \frac{1}{M} (\varepsilon_{m+1}^p - 2\varepsilon_m^p + \varepsilon_{m-1}^p) \\
 &= \varepsilon_m^p \left[ 1 + \frac{1}{M} (e^{i\beta\Delta x} - 2 + e^{-i\beta\Delta x}) \right] \\
 &= \varepsilon_m^p \left[ 1 + \frac{2}{M} (\cos\beta\Delta x - 1) \right]
 \end{aligned}$$

$$\text{i.e.: } G = \frac{\varepsilon_m^{p+1}}{\varepsilon_m^p} = 1 + \frac{2}{M} (\cos\beta\Delta x - 1) \quad (G: \text{amplification factor})$$

For the finite difference scheme to be stable:

$$\left| \frac{\varepsilon_m^{p+1}}{\varepsilon_m^p} \right| \leq 1$$

$$\text{therefore: } \left| 1 + \frac{2}{M} (\cos\beta\Delta x - 1) \right| \leq 1 \quad (\text{for any } \beta \text{ mode})$$

the maximum value occurs when  $\cos\beta\Delta x = -1$

$$\text{So: } \left| 1 - \frac{4}{M} \right| \leq 1 \quad \left( -1 \leq 1 - \frac{4}{M} \leq 1 \right)$$

$$\text{i.e.: } M \geq 2 \quad \left( M^{-1} = \frac{\Delta t}{\Delta x^2} \right)$$

$$\text{or: } \boxed{\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}}$$

The criteria for stable solution using simple explicit scheme.

\* Consider 1D implicit scheme of transient heat conduction equation:

$$\frac{T_m^{p+1} - T_m^p}{\Delta t} = \alpha \frac{T_{m+1}^{p+1} - 2T_m^{p+1} + T_{m-1}^{p+1}}{\Delta x^2}$$

i.e.,  $T_m^{p+1} - T_m^p = \frac{1}{M} (T_{m+1}^{p+1} - 2T_m^{p+1} + T_{m-1}^{p+1})$

or:  $T_m^{p+1} - \frac{1}{M} (T_{m+1}^{p+1} - 2T_m^{p+1} + T_{m-1}^{p+1}) = T_m^p$

stability analysis:

Let:  $\underline{\varepsilon_m^p = G^p e^{i\beta_{max} x}}$

so:  $\varepsilon_m^{p+1} - \frac{1}{M} (\varepsilon_{m+1}^{p+1} - 2\varepsilon_m^{p+1} + \varepsilon_{m-1}^{p+1}) = \varepsilon_m^p$

$$G^p e^{i\beta_{max} x} - \frac{1}{M} (G^{p+1} e^{i\beta_{max} x} - 2G^{p+1} e^{i\beta_{max} x} + G^{p+1} e^{i\beta_{max} x}) = G^p e^{i\beta_{max} x}$$

$$G^p e^{i\beta_{max} x} \left[ 1 - \frac{1}{M} (e^{i\beta_{max} x} - 2 + e^{-i\beta_{max} x}) \right] = G^p e^{i\beta_{max} x}$$

$$G \left[ 1 - \frac{2}{M} (\cos \beta_{max} x - 1) \right] = 1$$

$$G = \frac{1}{1 - \frac{2}{M} (\cos \beta_{max} x - 1)}$$

$$= \frac{1}{1 + \frac{2}{M} (1 - \cos \beta_{max} x)}$$

$G \leq 1$  for any  $\beta$  therefore:

the implicit scheme is unconditionally stable!